The asymptotic value in finite stochastic games

Miquel Oliu-Barton *

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Abstract

We provide a direct, elementary proof for the existence of $\lim_{\lambda \to 0} v_\lambda$, where v_λ is the value of λ-discounted finite two-person zero-sum stochastic game.

1 Introduction

Two-person zero-sum stochastic games were introduced by Shapley [4]. They are described by a 5-tuple $(\Omega, \mathcal{I}, \mathcal{J}, q, g)$, where Ω is a finite set of states, $\mathcal{I}$ and $\mathcal{J}$ are finite sets of actions, $g : \Omega \times \mathcal{I} \times \mathcal{J} \to [0,1]$ is the payoff, $q : \Omega \times \mathcal{I} \times \mathcal{J} \to \Delta(\Omega)$ the transition and, for any finite set X, $\Delta(X)$ denotes the set of probability distributions over X. The functions g and q are bilinearly extended to $\Omega \times \Delta(\mathcal{I}) \times \Delta(\mathcal{J})$. The stochastic game with initial state $\omega \in \Omega$ and discount factor $\lambda \in (0,1]$ is denoted by $\Gamma_\lambda(\omega)$ and is played as follows: at stage $m \geq 1$, knowing the current state ω_m, the players choose actions $(i_m, j_m) \in \mathcal{I} \times \mathcal{J}$; their choice produces a stage payoff $g(\omega_m, i_m, j_m)$ and influences the transition: a new state ω_{m+1} is chosen according to the probability distribution $q(\cdot|\omega_m, i_m, j_m)$. At the end of the game, player 1 receives $\sum_{m\geq 1} \lambda(1-\lambda)^{m-1} g(\omega_m, i_m, j_m)$ from player 2. The game $\Gamma_\lambda(\omega)$ has a value $v_\lambda(\omega)$, and $v_\lambda = (v_\lambda(\omega))_{\omega\in\Omega}$ is the unique fixed point of the so-called Shapley operator [4], i.e. $v_\lambda = \Phi(\lambda, v_\lambda)$, where for all $f \in \mathbb{R}^\Omega$:

$$\Phi(\lambda, f)(\omega) = \mathrm{val}_{(s,t)\in\Delta(\mathcal{I})\times\Delta(\mathcal{J})} \{\lambda g(\omega, s, t) + (1-\lambda)\mathbb{E}_{q(\cdot|\omega,s,t)}[f(\widetilde{\omega})]\}. \tag{1.1}$$

The Shapley operator provides optimal stationary strategies for both players. In particular, the result holds for any signalling structure on past actions. The existence of $\lim_{\lambda\to 0} v_\lambda$ was established by Bewley and Kohlberg [1], using Tarski-Seidenberg elimination theorem.

The purpose of this note is to provide a direct, self-contained proof for the existence of $\lim_{\lambda\to 0} v_\lambda$. The key idea is to represent the asymptotic behaviour of a sequence of strategies by a simpler object. Let $(x, y) \in \Delta(\mathcal{I})^\Omega \times \Delta(\mathcal{J})^\Omega$

*Combinatoire et Optimisation, IMJ, CNRS UMR 7586, Faculté de Mathématiques, Université P. et M. Curie Paris 6, Tour 15-16, 1er etage, 4 Place Jussieu, 75005 Paris and Université d'Orsay Paris-Sud https://sites.google.com/site/oliubarton

be a pair of stationary strategies. Every time the state $\omega \in \Omega$ is reached the next state is distributed according to $q(\cdot|\omega, x(\omega), y(\omega))$ and the stage payoff is $g(\omega, x(\omega), y(\omega))$. Thus, the sequence of states $(\omega_m)_m$ is a Markov chain with transition $Q = (q(\omega'|\omega, x(\omega), y(\omega))_{(\omega,\omega')\in\Omega^2}$ and the stage payoffs can be described by a vector $g = (g(\omega, x(\omega), y(\omega))_{\omega\in\Omega}$. For any initial state ω, the expected payoff induced by (x, y) in $\Gamma_\lambda(\omega)$ is given by

$$\gamma_\lambda(\omega, x, y) = \sum\nolimits_{\omega'\in\Omega} t_\lambda(\omega, \omega')g(\omega'),$$

where $t_\lambda(\omega, \omega') = \sum_{m\geq 1} \lambda(1-\lambda)^{m-1}Q^{m-1}(\omega, \omega')$ is the mean λ-discounted time spent in state ω'.

A key observation, due to Solan [5], is that $t_\lambda(\omega, \omega')$ can be written has a hitting time of an auxiliary Markov chain whose transitions are in the set $\{0, \lambda, ((1-\lambda)Q(\omega, \omega'))_{(\omega,\omega')\in\Omega^2}\}$. Thus, using a classical result from Friedlin and Wentzell for finite Markov chains, one deduces that $t_\lambda(\omega, \omega')$ is a rational fraction in the variables λ and $((1-\lambda)Q(\omega, \omega'))_{(\omega,\omega')\in\Omega^2}$, and that both polynomials in the numerator and denominator have nonnegative coefficients and are of degree at most $|\Omega|$. For a fixed y, a similar assertion is obtained for $\gamma_\lambda(\omega, x, y)$ as a function of the variables λ and $((1-\lambda)x^i(\omega))_{(\omega,i)\in\Omega\times\mathcal{I}}$. That is, $\gamma_\lambda(\omega, x, y)$ is a rational fraction in these variables. One can easily check that the monomials both in the numerator and denominator can then be written in the following form:

$$C(1-\lambda)^b\lambda^a \prod_{(\omega,i)\in\Omega\times\mathcal{I}} x^i(\omega)^{A(\omega,i)}, \tag{1.2}$$

where $C > 0$ depends on (y, ω) but not on (x, λ), $a, b \in \{0, \dots, |\Omega|\}$ and $A \in \{0, 1\}^{\Omega\times\mathcal{I}}$.

1.1 The asymptotic payoff

Consider now a sequence $(\lambda_n, x_n)_n$, where $\lambda_n \in (0, 1]$ is a discount factor and $x_n \in \Delta(\mathcal{I})^\Omega$ is a stationary strategy, for all $n \in \mathbb{N}$. $\gamma_{\lambda_n}(\omega, x_n, y)$, as n tends to infinity, for a fixed stationary strategy $y \in \Delta(\mathcal{J})^\Omega$.

Definition 1.1. *A sequence $(\lambda_n, x_n)_n$ in $(0, 1] \times \Delta(\mathcal{I})^\Omega$ is* regular *if $\lim_{n\to\infty} \lambda_n = 0$ and if for any two monomials of the form (1.2) their ratio converges in $[0, +\infty]$ as n tends to infinity.*[1]

Regular sequences can be characterized by a vector. Indeed, introduce a finite set:

$$\mathcal{M} := \{(A, a) \mid A \in \{-1, 0, 1\}^{\Omega\times\mathcal{I}},\ a \in \{-|\Omega|, \dots, 0, \dots, |\Omega|\}\}.$$

The sequence $(\lambda_n, x_n)_n$ is regular if for all $(A, a) \in \mathcal{M}$ the following limit

$$L[(\lambda_n, x_n)_n](A, a) := \lim_{n\to\infty} \lambda_n^a \prod_{(\omega,i)\in\Omega\times\mathcal{I}} x_n^i(\omega)^{A(\omega,i)} \tag{1.3}$$

exists in $[0, +\infty]$. The regularity of a sequence depends on the existence of finitely many limits. Thus, for any family $(x_\lambda)_{\lambda\in(0,1]}$ of stationary strategies there exists $(\lambda_n)_n$ such that $(\lambda_n, x_{\lambda_n})_n$ is regular.

[1] We use here the natural convention that $\frac{0}{0} = 0^0 = 1$ and $0^\beta = 0$, $0^{-\beta} = \frac{\beta}{0} = +\infty$, for all $\beta > 0$.

Proposition 1.1. *Let $y \in \Delta(\mathcal{J})^\Omega$ and $\omega \in \Omega$ be fixed. For any regular sequence $(\lambda_n, x_n)_n$, $\lim_{n\to\infty} \gamma_{\lambda_n}(\omega, x_n, y)$ exists and depends only on the vector $L[(\lambda_n, x_n)_n]$.*

Proof. Let $(\lambda_n, x_n)_n$ be regular and let $L = L[(\lambda_n, x_n)_n]$. We have already seen that the expected payoff induced by (x_n, y) in $\Gamma_{\lambda_n}(\omega)$ can be written as a rational fraction whose monomials are all of the form:

$$m_n := C(1-\lambda_n)^b \lambda_n^a \prod_{(\omega,i)\in\Omega\times\mathcal{I}} x_n^i(\omega)^{A(\omega,i)}, \tag{1.4}$$

that the ratio of any two monomials m_n and m_n' converges as $n \to \infty$, and that the limit is determined by L (and the constants $C, C' > 0$). Thus, one can use the vector L to define an order relation in the set of the monomials in $\gamma_{\lambda_n}(\omega, x_n, y)$ as follows: $m_n \preceq m_n'$ if and only if $\lim_{n\to\infty} m_n/m_n' \in [0, +\infty)$. The set is totally ordered. Dividing numerator and denominator by some maximal element m_n^*, and taking $n \to \infty$ we obtain that:

$$\lim_{n\to\infty} \gamma_{\lambda_n}(\omega, x_n, y) = \frac{\sum_{(A,a)\in\mathcal{M}^+} C(A,a) L(A - A^*, a - a^*)}{\sum_{(A,a)\in\mathcal{M}^+} C'(A,a) L(A - A^*, a - a^*)}, \tag{1.5}$$

where $\mathcal{M}^+ := \{(A,a) \mid A \in \{0,1\}^{\Omega\times\mathcal{I}},\ a \in \{0,\dots,|\Omega|\}\}$, and where the constants $C(A,a)$ and $C'(A,a)$ are nonnegative for all $(A,a) \in \mathcal{M}^+$. The maximality of m^* ensures that $L(A - A^*, a - a^*) \in [0, +\infty)$, for all $(A,a) \in \mathcal{M}^+$ and that not all are 0. The result follows. □ □

1.2 Canonical strategies

For any $\mathbf{c} = (\mathbf{c}(\omega, i))$ and $\mathbf{e} = (\mathbf{e}(\omega, i))$ in $\mathbb{R}_+^{\Omega\times I}$, we define a family of stationary strategies $(\mathbf{x}_\lambda)_\lambda$ as follows:

$$\mathbf{x}_\lambda^i(\omega) := \frac{\mathbf{c}(\omega,i)\lambda^{\mathbf{e}(\omega,i)}}{\sum_{i'\in\mathcal{I}} \mathbf{c}(\omega,i')\lambda^{\mathbf{e}(\omega,i')}}, \quad \forall(\omega,i) \in \Omega\times\mathcal{I},\ \forall\lambda \in (0,1]. \tag{1.6}$$

Assume, in addition, that $\sum_{i\in\mathcal{I},\ \mathbf{e}(\omega,i)=0} \mathbf{c}(\omega,i) = 1$ for all ω, so that

$$\mathbf{x}_\lambda^i(\omega) \sim_{\lambda\to 0} \mathbf{c}(\omega,i)\lambda^{\mathbf{e}(\omega,i)}, \quad \forall(\omega,i) \in \Omega\times\mathcal{I}. \tag{1.7}$$

The exponent determines the order of magnitude of the probability of playing the action i at state ω asymptotically; the coefficient $\mathbf{c}(\omega,i)$ its intensity.

Definition 1.2. *A family of strategies $(\mathbf{x}_\lambda)_{\lambda\in(0,1]}$ is* canonical *if it is induced by some $\mathbf{x} = (\mathbf{c}, \mathbf{e})$ in the following set:*

$$\mathbf{X} = \{(\mathbf{c},\mathbf{e}) \in (\mathbb{R}_+^* \times \mathbb{R}_+)^{\Omega\times\mathcal{I}} \mid \forall\omega \in \Omega,\ \sum_{i\in\mathcal{I},\ \mathbf{e}(\omega,i)=0} \mathbf{c}(\omega,i) = 1\}.$$

Note that the coefficients are taken strictly positive.

For all $(A,a) \in \mathcal{M}$ and $\mathbf{x} = (\mathbf{c},\mathbf{e}) \in \mathbf{X}$ the following limit exists:

$$L_{\mathbf{x}}(A,a) := \lim_{\lambda\to 0} \lambda^a \prod_{(\omega,i)} \mathbf{x}_\lambda^i(\omega)^{A(\omega,i)}. \tag{1.8}$$

Indeed, a direct consequence of (1.7) is that:

$$L_{\mathbf{x}}(A,a) = \lim_{\lambda\to 0} \lambda^{a+\sum_{(\omega,i)} A(\omega,i)\mathbf{e}(\omega,i)} \prod_{(\omega,i)} \mathbf{c}(\omega,i)^{A(\omega,i)},$$

where $\prod_{(\omega,i)} \mathbf{c}(\omega,i)^{A(\omega,i)} > 0$. Thus:

$$L_{\mathbf{x}}(A,a) \in \begin{cases} \{0\}, & \text{iff } a + \sum_{(\omega,i)} A(\omega,i)\mathbf{e}(\omega,i) > 0, \\ \{+\infty\}, & \text{iff } a + \sum_{(\omega,i)} A(\omega,i)\mathbf{e}(\omega,i) < 0, \\ (0,+\infty), & \text{iff } a + \sum_{(\omega,i)} A(\omega,i)\mathbf{e}(\omega,i) = 0. \end{cases} \tag{1.9}$$

Thus, for any $\mathbf{x} \in \mathbf{X}$ and any vanishing sequence $(\lambda_n)_n$ of discount factors, the sequence $(\lambda_n, \mathbf{x}_{\lambda_n})_n$ is regular. Moreover, $L_{\mathbf{x}} = L[(\lambda_n, \mathbf{x}_{\lambda_n})_n]$ for any such sequence.

2 Main results

2.1 Representation of a regular sequence by a canonical strategy

Fix some regular sequence $(\lambda_n, x_n)_n$ throughout this section and let $L = L[(\lambda_n, x_n)_n] \in [0,+\infty]^{\mathcal{M}}$ the vector defined in (1.3). Notice that L has many elementary properties:

$(P1)$ $L(0,0) = 1$ and, for all $(A,a) \neq 0$, $L(A,a) = +\infty$ if and only if $L(-A,-a) = 0$;

$(P2)$ For all $\mu \in \mathbb{R}$, $L(0,\mu) := \lim_{n\to\infty} \lambda_n^{\mu} = 0 \Leftrightarrow \mu > 0$ and $L(0,\mu) \in (0,+\infty) \Leftrightarrow \mu = 0$. In particular, $L(0,\mu) \in \{0,1,+\infty\}$ for all $\mu \in \mathbb{R}$;

$(P3)$ If $L(A,a) < +\infty$, $L(\mu A, \mu a) := \lim_{n\to\infty} \lambda_n^{\mu a} \prod_{(\omega,i)} x_n^i(\omega)^{\mu A(\omega,i)} = L(A,a)^{\mu}$;

$(P4)$ If $L(A,a) < +\infty$ and $L(B,b) < +\infty$, then $L(A+B, a+b) = L(A,a)L(B,b)$.

Proposition 2.1. *There exists $\mathbf{x} \in \mathbf{X}$ such that $L_{\mathbf{x}} = L$.*

Proof. Note that $\prod_{(\omega,i)} \mathbf{c}(\omega,i)^{A(\omega,i)} > 0$ for any $A \in \{-1,0,1\}^{\Omega\times I}$. Thus, from (1.9) and $(P1)$ one deduces the following necessary and sufficient conditions on the coefficients and the exponents $(\mathbf{c},\mathbf{e})$ of $\mathbf{x}$ for having $L_{\mathbf{x}} = L$:

$$\sum_{(\omega,i)} A(\omega,i)\mathbf{e}(\omega,i) + a > 0, \qquad \forall (A,a) \in \mathcal{M} \text{ s.t. } L(A,a) = 0, \tag{2.1}$$

$$\sum_{(\omega,i)} A(\omega,i)\mathbf{e}(\omega,i) + a = 0, \quad \forall (A,a) \in \mathcal{M} \text{ s.t. } L(A,a) \in (0,+\infty), \tag{2.2}$$

$$\prod_{(\omega,i)} \mathbf{c}(\omega,i)^{A(\omega,i)} = L(A,a), \quad \forall (A,a) \in \mathcal{M} \text{ s.t. } L(A,a) \in (0,+\infty). \tag{2.3}$$

Notation: Let $\mathcal{L}_0 := \{(A,a) \in \mathcal{M} \mid L(A,a) = 0\}$ and $\mathcal{L}_+ := \{(A,a) \in \mathcal{M} \mid L(A,a) \in (0,+\infty)\}$. Put $\mathcal{L} := \mathcal{L}_0 \cup \mathcal{L}_+$.

Solving for the exponents. Let us prove that the system (2.1)-(2.2) has a solution. One and only one of the systems (2.1)-(2.2) and (2.4)-(2.5)-(2.6) is consistent (see Mertens, Sorin and Zamir [3], part A, page 28):

$$\sum_{(A,a)\in\mathcal{L}} \mu(A,a)A = 0, \quad \mu_{|\mathcal{L}_0} \geq 0, \tag{2.4}$$

$$-\sum_{(A,a)\in\mathcal{L}} \mu(A,a)a \geq 0, \tag{2.5}$$

$$-\sum_{(A,a)\in\mathcal{L}} \mu(A,a)a + \sum_{(A,a)\in\mathcal{L}_0} \mu(A,a) > 0, \tag{2.6}$$

Let us prove that the system (2.4)-(2.5)-(2.6), with unknowns $\mu = (\mu(A,a))_{(A,a)} \in \mathbb{R}^{\mathcal{L}}$, is inconsistent. In (2.4), $\mu_{|\mathcal{L}_0} := (\mu(A,a))_{(A,a)\in\mathcal{L}_0}$ denotes the restriction of μ to $\mathcal{L}_0$. Assume (2.4). On the one hand, by $(P3)$-$(P4)$, for all $\mu \in \mathbb{R}^{\mathcal{L}}$:

$$\prod_{(A,a)\in\mathcal{L}_+} L(A,a)^{\mu(A,a)} = L\left(\sum\nolimits_{(A,a)\in\mathcal{L}_+} \mu(A,a)A, \sum\nolimits_{(A,a)\in\mathcal{L}_+} \mu(A,a)a\right) \in (0,+\infty) \qquad (2.7)$$

On the other hand, by $(P3)$-$(P4)$, for all $\mu \in \mathbb{R}^{\mathcal{L}}$ such that $\mu_{|\mathcal{L}_0} \geq 0$ one has:

$$\prod_{(A,a)\in\mathcal{L}_0} L(A,a)^{\mu(A,a)} = L\Big(\sum_{(A,a)\in\mathcal{L}_0} \mu(A,a)A, \sum_{(A,a)\in\mathcal{L}_0} \mu(A,a)a\Big) = \begin{cases} 1 & \text{if } \mu_{|\mathcal{L}_0} = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (2.8)$$

Multiplying (2.7) and (2.8) yields, by assumption (2.4) :

$$L\left(0, \sum\nolimits_{(A,a)\in\mathcal{L}} \mu(A,a)a\right) \in \begin{cases} (0,+\infty) & \text{if } \mu_{|\mathcal{L}_0} = 0, \\ \{0\} & \text{otherwise.} \end{cases} \qquad (2.9)$$

By $(P2)$, the first case implies $\sum_{(A,a)\in\mathcal{L}} \mu(A,a)a = 0$, which contradicts (2.6), and the second case implies $\sum_{(A,a)\in\mathcal{L}} \mu(A,a)a > 0$, which contradicts (2.5). The system (2.4)-(2.5)-(2.6) being inconsistent, the existence of a solution to (2.1)-(2.2) in $\mathbb{R}^{\Omega\times\mathcal{I}}$ follows. The boundedness of $x_n(\omega,i)$ implies that $L((0,\ldots,1^{(\omega,i)},\ldots,0),0) < +\infty$, so that $\mathbf{e}(\omega,i) \geq 0$ by (2.1) and (2.2).

Solving for the coefficients. Taking the logarithm in (2.3) yields:

$$\sum\nolimits_{(\omega,i)} A(\omega,i)\ln \mathbf{c}(\omega,i) = \ln(L(A,a)), \quad \forall (A,a) \in \mathcal{L}_+, \qquad (2.10)$$

which is a linear system in $\mathbf{d} = (\ln \mathbf{c}(\omega,i))_{(\omega,i)} \in \mathbb{R}^{\Omega\times\mathcal{I}}$. As before, one and only one of the systems (2.10) and (2.11) is consistent:

$$\sum\nolimits_{(A,a)\in\mathcal{L}_+} \mu(A,a)A = 0, \quad \sum\nolimits_{(A,a)\in\mathcal{L}_+} \mu(A,a)\ln(L(A,a)) > 0. \qquad (2.11)$$

Let us prove that the system (2.11), with unkowns $\mu = (\mu(A,a))_{(A,a)} \in \mathbb{R}^{\mathcal{L}_+}$, is inconsistent. Suppose that $\sum_{(A,a)\in\mathcal{L}_+} \mu(A,a)A = 0$. Then, by $(P3)$-$(P4)$:

$$\prod\nolimits_{(A,a)\in\mathcal{L}_+} L(A,a)^{\mu(A,a)} = L\left(0, \sum\nolimits_{(A,a)\in\mathcal{L}_+} \mu(A,a)a\right) \in (0,+\infty).$$

By $(P2)$, this implies $\sum_{(A,a)\in\mathcal{L}_+} \mu(A,a)a = 0$ and, a fortiori, $\prod_{(A,a)\in\mathcal{L}_+} L(A,a)^{\mu(A,a)} = 1$, so that (2.11) fails. Consequently, there exists $\mathbf{c} = (\exp(\mathbf{d}(\omega,i)) \in (\mathbb{R}_+^*)^{\Omega\times I}$ satisfying (2.3). □

2.2 Convergence of the discounted values

Theorem 2.1. *The limit of* $(v_\lambda)_\lambda$*, as* λ *tends to* 0*, exists. Moreover, there exists* $\mathbf{x} \in \mathbf{X}$ *such that* $(\mathbf{x}_\lambda)_\lambda$ *is* asymptotically optimal, *i.e. for all* $\varepsilon > 0$*, there exists* $\lambda_0 \in (0,1]$ *such that:*

$$\gamma_\lambda(\omega, \mathbf{x}_\lambda, y) \geq \lim\nolimits_{\lambda\to 0} v_\lambda(\omega) - \varepsilon, \quad \forall \omega \in \Omega,\ \forall y \in \Delta(\mathcal{J})^\Omega,\ \forall \lambda \in (0,\lambda_0).$$

Proof. Let $\omega \in \Omega$ be fixed. Let $(x_\lambda)_{\lambda>0}$ be a family of optimal stationary strategies in $(\Gamma_\lambda(\omega))_{\lambda>0}$ and let $(\lambda_n)_n$ be a sequence of discount factors such that $\lim_{n\to\infty} v_{\lambda_n}(\omega) = \limsup_{\lambda\to 0} v_\lambda(\omega)$. The optimality of x_{λ_n} implies that $\gamma_{\lambda_n}(\omega, x_{\lambda_n}, j) \geq v_{\lambda_n}(\omega)$, for all $j \in \mathcal{J}^\Omega$. Indeed, against a stationary strategy of player 1, player 2 faces a Markov decision process. Thus, player 2 has a pure stationary best reply. Up to some subsequence, $(\lambda_n, x_{\lambda_n})_n$ is regular. By Proposition 2.1, there exists $\mathbf{x} \in \mathbf{X}$ such that $L_{\mathbf{x}} = L[(\lambda_n, x_{\lambda_n})_n]$. Thus, by Proposition 1.1,

$$\lim_{n\to\infty} \gamma_{\lambda_n}(\omega, x_{\lambda_n}, j) = \lim_{n\to\infty} \gamma_{\lambda_n}(\omega, \mathbf{x}_{\lambda_n}, j), \quad \forall j \in \mathcal{J}^\Omega.$$

On the other hand, the limit $\lim_{\lambda\to 0} \gamma_\lambda(\omega, \mathbf{x}_\lambda, j)$ exists. Consequently:

$$\lim_{\lambda\to 0} \gamma_\lambda(\omega, \mathbf{x}_\lambda, j) = \lim_{n\to\infty} \gamma_{\lambda_n}(\omega, x_{\lambda_n}, j) \geq \limsup_{\lambda\to 0} v_\lambda(\omega), \quad \forall j \in \mathcal{J}^\Omega. \tag{2.12}$$

It follows that for all $\varepsilon > 0$ there exists $\lambda_0 \in (0,1]$ such that:

$$\min_{j\in\mathcal{J}^\Omega} \gamma_\lambda(\omega, \mathbf{x}_\lambda, j) \geq \limsup_{\lambda\to 0} v_\lambda(\omega) - \varepsilon, \quad \forall \lambda \in (0, \lambda_0). \tag{2.13}$$

The latter implies that $v_\lambda(\omega) \geq \limsup_{\lambda\to 0} v_\lambda(\omega) - \varepsilon$, for all $\lambda \in (0, \lambda_0)$, and the existence of $\lim_{\lambda\to 0} v_\lambda$ follows by taking the lim inf. The canonical strategy $\mathbf{x}$ has the desired property. ☐

2.3 Concluding remarks

(1) Consider an infinitely repeated stochastic game where the past actions are observed. The existence of the uniform value is due to Mertens and Neyman [2] and relies on the following result:

Theorem 2.2. *Let $f : (0,1) \to \mathbb{R}^\Omega$ be a function such that:*

(a) $\|f_\lambda - f_{\lambda'}\| \leq \int_\lambda^{\lambda'} \varphi(x)dx$, for all $0 < \lambda < \lambda' < 1$ and for some $\varphi \in L^1\big((0,1], \mathbb{R}_+\big)$;

(b) There exists $\lambda_0 > 0$ such that $\Phi(\lambda, f_\lambda) \geq f_\lambda$, for all $\lambda \in (0, \lambda_0)$.[2]

Then, player 1 can guarantee $\lim_{\lambda\to 0} f_\lambda$ in Γ_∞.

One can use Theorem 2.1 to prove the existence of the uniform value. Indeed, for any $x \in \Delta(\mathcal{I})^\Omega$, $\omega \in \Omega$ and $\lambda \in (0,1]$, let $w_\lambda^x(\omega) := \min_{j\in\mathcal{J}^\Omega} \gamma_\lambda(\omega, x, j)$ be the payoff guaranteed by x in $\Gamma_\lambda(\omega)$. One can check that $w_\lambda^x \leq \Phi(\lambda, w_\lambda^x)$, for all $\lambda \in (0,1]$. Besides, for any $\mathbf{x} \in \mathbf{X}$, the functions $(\lambda \mapsto w_\lambda^{\mathbf{x}_\lambda}(\omega))_{\omega\in\Omega}$ are of bounded variation, so that player 1 can guarantee $\lim_{\lambda\to 0} w_\lambda^{\mathbf{x}_\lambda}$ for any $\mathbf{x} \in \mathbf{X}$ by Theorem 2.2. In particular, if $(\mathbf{x}_\lambda)_\lambda$ is asymptotically optimal, player 1 can guarantee $\lim_{\lambda\to 0} v_\lambda$.

(2) The existence of an $\mathbf{x} \in \mathbf{X}$ such that $(\mathbf{x}_\lambda)_\lambda$ is asymptotically optimal was already noticed by Solan and Vieille [6]. The result was deduced from the semi-algebraicity of $\lambda \mapsto v_\lambda$, obtained in [1] using Tarski-Seidenberg elimination theorem.

[2] Φ is the Shapley operator, defined in (1.1).

(3) In the system (2.1)-(2.2) for the exponents (first part of the proof of Proposition 2.1) note that all the entries of A are in $\{-1, 0, 1\}$. This implies the existence of a solution having all its coordinates in $\{0, 1/N, 2/N, \dots\}$, for some $N \leq |\Omega||\mathcal{I}|^{\sqrt{|\Omega||\mathcal{I}|}}$.

(4) Our approach fails without the finiteness assumption on $\mathcal{I}$, $\mathcal{J}$ and Ω. A recent example where $\mathcal{I}$ and $\mathcal{J}$ are compact, q is continuous, g is independent of the actions and the family $(v_\lambda)_\lambda$ does not converge is due to Vigeral [7].

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